Zilch Expected Scores*

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Abstract

Probability analysis of the game Zilch [1]. (Incomplete)

1 Expected Score Formula

Here is the recurrence relation for $E(y)$, the Expected Score for the entire turn starting with rolling y dice (includes subsequent throws), where $S(y,x)$ is the average Score for taking x best dice from a roll of y dice, $Z(x)$ is the probability of getting a zilch with x dice, and $EZ(x)$ is the probability of getting a zilch in any future throw starting with x dice and playing the optimal strategy.

$$E(y) = \max \begin{cases} \max_x S(y,x) \\ \max_x \{S(y,x)(1 - EZ(y-x)) + E(y-x)(1 - Z(y))\} \end{cases} \tag{1}$$

$$E(0) = E(6) \tag{2}$$

1.1 Short Summary

The Expected Score starting from any single throw is maximized by selecting the best strategy that maximizes the score taken now plus Expected Score for the next throw. This is clearly a recursive relation.

The max function selects the best bank strategy (top) vs the best reroll strategy (bottom). The bank strategy should be straightforward; we just take x dice to bank, maximizing the score $S(y,x)$, and we are finished.

The reroll strategy is more tricky. We consider all possibilities of choosing x dice and rerolling the other $y-x$ dice. A reroll is only allowed if the original y dice were not a zilch, so we must multiply $E(y-x)$ by the probability of not zilching $1 - Z(y)$. Additionally, taken score $S(y,x)$ is nullified if a zilch

*http://zalbee.intricus.net/zilch.pdf

is encountered in *any* subsequent throw (not just the next); therefore we need another function EZ, to represent total future Expected Zilch probability.

Eq. 2 takes into account "free rolls," by allowing the case $x = y$ (all dice are taken) to give a valid value of $E(y - x) = E(0)$, i.e. the same as rolling 6 dice again.

1.2 From the beginning

First we define `score(dice)` as a function on actual dice values as opposed to $S(y, x)$ which is only a function on a number of dice. Similarly,

`score(dice)`	Score for taking dice
`expect(dice)`	Expected Score after rolling dice
`zilch(dice)`	Zilch Prob. (0 or 1)
`expectzilch(dice)`	Expected Zilch Prob.

Let's say you roll {1,2,2,4,4,5}. There are two scoring options: 1 and 5. Let's calculate the expected score.

$$\texttt{expect}(\{1,2,2,4,4,5\}) = \max \begin{cases} \texttt{score}(\{1\}) \\ \texttt{score}(\{5\}) \\ \texttt{score}(\{1,5\}) \\ \texttt{score}(\{1\}) + E(5) \\ \texttt{score}(\{5\}) + E(5) \\ \texttt{score}(\{1,5\}) + E(4) \end{cases} \tag{3}$$

We must use $E(5)$ instead of `expect(dice)` function because we don't know what the result of the next roll will be. We instead use E which gives an average expected value across all possibilities.

One more thing is missing: if any subsequent roll is a zilch, we get zero score, whereas this formula is purely additive. Therefore, we should reduce `score` by multiplying against the probability of not zilching.

$$\texttt{expect}(\{1,2,2,4,4,5\}) = \max \begin{cases} \texttt{score}(\{1\}) \\ \texttt{score}(\{5\}) \\ \texttt{score}(\{1,5\}) \\ \texttt{score}(\{1\})(1 - EZ(5)) + E(5) \\ \texttt{score}(\{5\})(1 - EZ(5)) + E(5) \\ \texttt{score}(\{1,5\})(1 - EZ(4)) + E(4) \end{cases} \tag{4}$$

Does this account for zilches on the first roll? Let's say we want the expected value of {2,3,4,6}.

$$\texttt{expect}(\{2,3,4,6\}) = 0$$

There should be no entries in the max function, because there are no scoring options where $\texttt{score} > 0$.

Example: we combine the above cases, to show that this formula gives the correct answer in the case where we first roll {1,2,2,4,4,5}, keep {1,5}, then reroll {2,3,4,6}:

$$\begin{aligned}\text{`}\texttt{expect}\text{'}(\{1,2,2,4,4,5\}) &= \texttt{score}(\{1,5\})(1-\texttt{expectzilch}(\{2,3,4,6\})) + \texttt{expect}(\{2,3,4,6\}) \\ &= (150)(1-1)+0 \\ &= 0\end{aligned} \tag{5}$$

We then generalize Eq. 4:

$$E(y) = \max\begin{cases}\max_x S(y,x) \\ \max_x \{S(y,x)(1-EZ(y-x)) + E(y-x)\}\end{cases} \tag{6}$$

When we generalize `score` to S, S is an average of all scores, including 0's (zilches). We can only do an extra roll if the original roll is not a zilch, so we must multiply $E(y-x)$ by the probability of not zilching $1-Z(y)$. This brings us to the final Equation 1:

$$E(y) = \max\begin{cases}\max_x S(y,x) \\ \max_x \{S(y,x)(1-EZ(y-x)) + E(y-x)(1-Z(y))\}\end{cases}$$

2 Tractability

If the recurrence relation for E is correct and solvable, we should end up with 6 values, $E(1)$ – $E(6)$. The dependency on EZ (which itself depends on E) complicates things. Z has been computed before [2], and S should be computable in the same way.

3 Zilch Probability

This one is easy and has already been calculated by Leadhyena [2].

$$Z(1) = 2/3 \tag{7}$$
$$Z(2) = 4/9 \tag{8}$$
$$Z(3) = 0.27777... \tag{9}$$
$$Z(4) = 0.15740740... \tag{10}$$
$$Z(5) = 0.07716049382716049 \tag{11}$$
$$Z(0) = Z(6) = 0 \tag{12}$$

4 Score for Taking x Dice

Let's define $S(y, x)$ more precisely as the average, across all y-dice throws, of the score of the best x dice in each throw.

$$S(y, x) = \texttt{avg}_{y\texttt{-throws}} \{\max \{\texttt{score}(D) \ : \ |D| = x, D \subseteq \texttt{throw}\}\} \tag{13}$$

Example with base case $x = 1$:

- For rolling 1 die, we have 1/6 chance for 100 points (roll 1), 1/6 chance for 50 points (roll 5). $S(1, 1) = 25$.

- For rolling 6 dice, we have $1 - (5/6)^6 = 0.665$ chance for 100 points (roll at least one 1), $(1 - (4/5)^6)(5/6)^6 = 0.247$ chance for 50 points (roll at least one 5 and no 1's). $S(6, 1) = 78.8655$.

We could calculate the rest with math, but it's easier to write a program to brute-force the values. I have done that, by extending Leadhyena's program, and come up with the following values of S:

$S(y, x)$	$x = 6$	5	4	3	2	1
$y = 1$						25
$y = 2$					50	43.056
$y = 3$				86.806	72.917	56.25
$y = 4$			143.519	138.657	92.978	66.011
$y = 5$		225.791	223.058	204.090	110.089	73.322
$y = 6$	427.437	334.475	323.002	278.142	124.468	78.8655

5 Expected Zilch Probability Including Future Rolls

$$EZ(x) = \begin{cases} Z(x) + (1 - Z(x))(EZ(r)) & \text{if } E(x) \text{ chose to roll } r \\ Z(x) & \text{if } E(x) \text{ chose to bank} \end{cases} \tag{14}$$

We now have somewhat of a paradox, as we need the decision from $E(x)$ to solve $EZ(x)$, but we need $EZ(x - y)$ to solve $E(x)$. The key now will be to find a workable base case, but this is unfortunately difficult due to the wraparound caused by "free rolls":

$$EZ(0) = EZ(6) \tag{15}$$

References

[1] Gaby, *Zilch*, http://blog.playr.co.uk/2008/11/new-game-zilch.html (2008)

[2] Leadhyena Inrandomtan, *Study of the game Zilch part 1*, Viviomancy Blog, http://viviomancy.blogspot.com/2008/11/study-of-game-zilch-part-1.html (2008)